B.Sc III
Contd:-

Problem Based on Lagrange's Undetermined Multipliers

Prob. 3

Find the maximum and minimum value of $x^2+y^2+z^2$ subject to the conditions

$$ax+by+cz=1$$
$$a'x+b'y+c'z=1$$

Solution :⇒ Let $u = x^2+y^2+z^2$ —— (A)

for maximum or minimum values of u, we have $du=0$

Diff (1) partially

$$du = 2x\,dx + 2y\,dy + 2z\,dz = 0$$
$$= x\,dx + y\,dy + z\,dz = 0 \qquad \text{—— (I)}$$

$$\because ax+by+cz-1=0$$
$$\therefore a\,dx + b\,dy + c\,dz = 0 \qquad \text{—— (II)}$$
$$\because a'x + b'y + c'z - 1 = 0$$
$$\Rightarrow a'dx + b'dy + c'dz = 0 \qquad \text{—— (III)}$$

Now $x\,dx + y\,dy + z\,dz + \lambda_1(a\,dx + b\,dy + c\,dz) + \lambda_2(a'dx + b'dy + c'dz) = 0$

$$\Rightarrow (x + a\lambda_1 + a'\lambda_2)dx + (y + \lambda_1 b + \lambda_2 b')dy + (z + \lambda_1 c + \lambda_2 c')dz = 0$$

Now equaling to zero, the co-efficients of dx, dy, dz.

we get

$$x + a\lambda_1 + a'\lambda_2 = 0 \qquad \text{—— (IV)}$$
$$y + b\lambda_1 + b'\lambda_2 = 0 \qquad \text{—— (V)}$$
$$z + c\lambda_1 + c'\lambda_2 = 0 \qquad \text{—— (VI)}$$

$x \times$ (IV) $+ y \times$ (V) $+ z \times$ (VI)

$$(x^2+y^2+z^2) + \lambda_1(ax+by+cz) + \lambda_2(a'x+b'y+c'z) = 0$$
$$\Rightarrow u + \lambda_1 + \lambda_2 = 0 \qquad \text{—— (VII)}$$

Again $a \times$ (IV) $+ b \times$ (V) $+ c \times$ (VI) $\cancel{\Sigma a^2 = a^2+b^2}+$

$$(ax+by+cz) + \lambda_1(\Sigma a^2) + \lambda_2\,\Sigma aa' = 0$$

$$1 + \lambda_1 \Sigma a^2 + \lambda_2 \Sigma a a' = 0 \quad \text{—— (VIII)}$$

Again ~~Multiplying~~ $(IV) \times a' + (V) \times b' + (VI) \times c'$

we have

$$(a'x + b'y + c'z) + \lambda_1 \Sigma a a' + \lambda_2 \Sigma a'^2 \quad \text{—— (IX)}$$

Now eliminating λ_1 and λ_2 from

$$u + \lambda_1 + \lambda_2 = 0$$

$$1 + \lambda_1 \Sigma a^2 + \lambda_2 \Sigma a a' = 0$$

$$1 + \lambda_1 \Sigma a a' + \lambda_2 \Sigma a'^2 = 0$$

we get

$$\begin{vmatrix} u & 1 & 1 \\ 1 & \Sigma a^2 & \Sigma a a' \\ 1 & \Sigma a a' & \Sigma a'^2 \end{vmatrix} = 0$$

it gives the maximum or minimum value of u.

Problem 4. Find the Extreme values of the function

$$u = x^2 + y^2 + z^2$$

Subject to two equations of condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$lx + my + nz = 0$$

This question may be asked as.

Find the length of the axes of the section of Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ by the Plane.}$$

$$lx+my+nz=0$$

Here. $u=x^2+y^2+z^2$ — (I)
under condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ — (II)
and $lx+my+nz=0$ — (III)

Let $u = x^2+y^2+z^2$ ————— (I)

Diff. Partially w.r.t. x, y, z.

$$du = 2x\,dx + 2y\,dy + 2z\,dz.$$

Since u is Extreme value

$\Rightarrow du = 0$

$\Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 0$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$ ——— (IV)

Similarly Diff. (II) and (III) Partially.

we have

$$\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz=0$$

$$\Rightarrow \frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0 \quad \text{——— (V)}$$

Diff. Partially (III)

we have $l\,dx + m\,dy + n\,dz = 0$ ——— (VI)

Now (IV) $+ \lambda_1$ (V) $+ \lambda_2$ (VI)

we have

$$\left(x+\frac{\lambda_1}{a^2}+\lambda_2 l\right)dx+\left(y+\frac{\lambda_1}{b^2}+\lambda_2 m\right)dy+\left(z+\frac{\lambda_1}{c^2}+\lambda_2 n\right)dz=0$$

Since dx, dy, dz are independent to each other, therefore

$$x+\frac{\lambda_1}{a^2}+\lambda_2 l=0 \quad \text{——— (VII)}$$

$$y+\frac{\lambda_1}{b^2}+\lambda_2 m=0 \quad \text{——— (VIII)}$$

$$z+\frac{\lambda_1}{c^2}+\lambda_2 n=0 \quad \text{——— (IX)}$$

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Now we ~~may~~ require to ~~elt~~ eliminate λ_1 and λ_2 from equations

$$x + \lambda_1 \frac{x}{a^2} + \lambda_2 l = 0$$

$$y + \lambda_1 \frac{y}{b^2} + \lambda_2 m = 0$$

$$z + \lambda_1 \frac{z}{c^2} + \lambda_2 n = 0$$

Now

$$\therefore x\left(x + \lambda_1 \frac{x}{a^2} + \lambda_2 l\right) + y\left[y + \lambda_1 \frac{y}{b^2} + \lambda_2 m\right] + z\left[z + \frac{\lambda_1 z}{c^2} + \lambda_2 n\right] = 0$$

$$\Rightarrow (x^2 + y^2 + z^2) + \lambda_1\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \lambda_2 (lx + my + nz) = 0$$

$$\Rightarrow u + \lambda_1 + \lambda(0) = 0$$

$$\Rightarrow u = -\lambda_1$$

Putting $\lambda_1 = -u$ in

$$x + \lambda_1\left(\frac{x}{a^2}\right) + \lambda_2 l = 0$$

$$x - \frac{ux}{a^2} + \lambda_2 l = 0$$

$$x\left(1 - \frac{u}{a^2}\right) = -\lambda_2 l$$

$$\Rightarrow x\left(\frac{u}{a^2} - 1\right) = \lambda_2 l$$

$$\therefore x\left(\frac{u - a^2}{a^2}\right) = \lambda_2 l$$

$$\therefore x = \frac{\lambda_2 a^2 l}{u - a^2}$$

Similary from (VIII) and (IX)

$$y = \frac{\lambda_2 b^2 m}{u - b^2}$$

$$z = \frac{\lambda_2 c^2 n}{u - c^2}$$

But $lx + my + nz = 0$

Therefore $$\frac{\lambda_2 a^2 l^2}{u - a^2} + \lambda_2 \frac{b^2 m^2}{u - b^2} + \lambda_2 \frac{c^2 n^2}{u - c^2} = 0$$

$$\Rightarrow \frac{a^2 l^2}{u - a^2} + \frac{b^2 m^2}{u - b^2} + \frac{c^2 n^2}{u - c^2} = 0$$

Since $\lambda_2 \neq 0$

It is quadratic in u,
Value of u obtained from this
gives the extreme value.

Ex. Problem

Prove that of all the rectangular Parallelopiped of the same volume the Cubic has the least surface.

Solution — Say x, y, z are the length of three conterminous edges of the rectangular Parallelopiped

that volume $V = xyz = C$
$= xyz - C = 0$

Let $g(x,y,z) = xyz = C$

The surface $S = 2(xy + yz + zx)$
Here we have to find the minimum value of $S = 2(xy + yz + zx)$ ——— (I)

subject to the condition
$g(x,y,z) = xyz - C = 0$ ——— (II)

for maximum or minimum.

$dS = 0$

$\Rightarrow (x\,dy + y\,dx) + (y\,dz + z\,dy) + (z\,dx + x\,dz) = 0$

$\Rightarrow (y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$ ——— (III)

Differentiating partially (II)

$yz\,dx + zx\,dy + xy\,dz = 0$ ——— (IV)

Now we have

$$(y+z)\,dx + (z+x)\,dy + (x+y)\,dz + \lambda\,(yz\,dx + zx\,dy + xy\,dz) = 0.$$

$$\Rightarrow y+z+\lambda yz = 0$$

$$z+x+\lambda zx = 0$$

$$x+y+\lambda xy = 0$$

$$\therefore -\lambda = \frac{y+z}{yz} = \frac{z+x}{zx} = \frac{x+y}{xy}.$$

$$\frac{1}{y}+\frac{1}{z} = \frac{1}{z}+\frac{1}{x} = \frac{1}{x}+\frac{1}{y}$$

From $\frac{1}{y}+\frac{1}{z} = \frac{1}{z}+\frac{1}{x}$

$\Rightarrow \frac{1}{y} = \frac{1}{x}$ Also $\frac{1}{z}+\frac{1}{x} = \frac{1}{x}+\frac{1}{y}$

$\Rightarrow x = y.$ $\Rightarrow y = z$

$\therefore$ thus we find here for maximum or minimum value of S.

$$x = y = z$$

Now it remains to show whether S is maximum or minimum.

Differentiating $xyz = c$ partially, regarding z as a function of x and y we get.

$$y\left[x\frac{\partial z}{\partial x} + z\right] = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x}$$

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Similarly $\frac{\partial z}{\partial y} = -\frac{z}{y}$

Now from $\frac{\partial s}{\partial x} = 2\left[y + y\frac{\partial z}{\partial x} + \left(z\cdot 1 + x\frac{\partial z}{\partial x}\right)\right]$

$$= 2\left[y + z + (y+x)\frac{\partial z}{\partial x}\right]$$

$$= 2\left[y + z + (y+x)(-z/x)\right]$$

$$= 2\left[y + \cancel{z} - \frac{yz}{x} - \cancel{z}\right] = 2y - \frac{2yz}{x}$$

Similarly $\frac{\partial s}{\partial y} = 2x - \frac{2xz}{y}$

$$r = \frac{\partial^2 s}{\partial x^2} = 0 - 2y\left[\frac{1}{x}\frac{\partial z}{\partial x} + z\left(-\frac{1}{x^2}\right)\right]$$

$$= -2y\left[\frac{1}{x}\frac{\partial z}{\partial x} - \frac{z}{x^2}\right]$$

$$= -2y\left[\frac{1}{x}\left(-\frac{z}{x}\right) - \frac{z}{x^2}\right]$$

$$= -2y\left[-\frac{2z}{x^2}\right]$$

$$= \frac{4yz}{x^2} = 4 \quad \because x = y = z.$$

$$s = \frac{\partial^2 s}{\partial x \partial y} = 2 - \frac{2}{y}\left\{x\frac{\partial z}{\partial x} + z\right\}$$

$$= 2 - \frac{2}{y}\left\{x\left(-\frac{z}{x}\right) + z\right\}$$

$$= 2 - \frac{2}{y}(-z + z) = 2$$

$$t = \frac{\partial^2 s}{\partial y^2} = 0 - 2x\left\{\frac{1}{y}\frac{\partial z}{\partial y} + z\left(-\frac{1}{y^2}\right)\right\}$$

$$= -2x\left\{\frac{1}{y}\left(-\frac{z}{y}\right) - \frac{z}{y^2}\right\} = \frac{4xz}{y^2} = 4 \quad \because x = y = z$$

When $x = y = z$ i.e Parallelopiped is a cubic

Here $r > 0$ $rt - s^2 = 12 > 0$ $\Rightarrow$ Hence the surface is least